# Study material
# On
# Mechanics (Analytical Statics)

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| **Course Name:** | **UG, Semester-VI** |
| **Subject:** | **Mathematics** |
| **Paper:** | **Mechanics (Analytical Statics)** |

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**Subject: Mathematics** **Paper: Mechanics (Analytical Statics)**

## Assignment cum Important problems on Friction

1. Define : Sliding and Rolling frictions, Angle of friction, Cone of friction, Coefficient of friction
2. State the laws of statical friction.
3. A body is placed on a rough inclined plane to the horizon at an angle greater than the angle of friction and is supported by a force. Show that the force required to move the body up the plane will be least when it is applied in a direction making with the plane an angle equal to the angle of friction.
4. A rough wire which has the shape of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is placed with its $x$ axis vertical and $y$ axis horizontal. If $\mu$ be the coefficient of friction, find the depth below the highest point of the position of equilibrium of a bead which rest on the wire.
5. Prove that the angle of friction satisfies the condition of equilibrium of a particle on a rough curve $f(x,y)=0$ under system of coplanar forces $(X,Y)$, $\cos\lambda \leq \frac{Xf_x+Yf_y}{\sqrt{X^2+Y^2}\sqrt{{f_x}^2+{f_y}^2}}$.
6. A uniform ladder of weight $W$ rests on a rough horizontal ground and against a smooth vertical wall, inclined at angle $\alpha$ to the horizon. Find the condition that a man of weight $W'$ can climb to the top of the ladder without the ladder slipping, in terms of $,W',\mu$ .
7. A sphere of radius $a$ and whose c.g. is at a distance$c$ from the centre resting in limiting equilibrium on a rough plane inclined at an angle $\alpha$ to the horizon. Show that it may be turned through an angle $2\cos^{-1}(\frac{a\sin\alpha}{c})$ and still be in equilibrium.
8. A solid hemisphere of weight $W$ rests in limiting equilibrium with its curved surface on a rough inclined plane, and the plane face is horizontal by a weight $P$ attached at a point in the rim. Find the value of the coefficient of friction.
9. A semicircular disc rests in a vertical plane with its curved edge on a horizontal and an equally rough vertical plane, the coefficient of friction being $\mu$. Find the greatest angle that the bounding diameter can make with the horizontal plane.
10. A perfectly rough plane is inclined at an angle $\alpha$ to the horizon; find the least eccentricity of the ellipse which can rest on the plane.
11. A solid homogeneous hemisphere rests on a rough horizontal plane and against another rough vertical wall. Find the least angle that the base of the hemisphere can make with the vertical. Also, discuss the different positions of equilibrium.

## Assignment cum Important problems on Virtual Work (V. W)

1. State and prove the principle of virtual work for a system of coplanar forces.
2. State and prove the principle of virtual work for a particle acted upon a system of coplanar forces.
3. Mention the forces which are appear and do not appear in the equation of V. W.
4. A string of length $l$ forms the shorter diagonal of a rhombus formed by four uniform rods, each of length $a$ and weight $W$ which are hinged together. If one of rods be supported in a horizontal position, find the tension of the string.
5. Four equal rods each of weight $W$ form a rhombus$ABCD$ with smooth hinges at the joints. The frame is suspended by the end $A$ and a weight $W'$ is attached at $C$. A stiffening rod of negligible weight joins the middle points of $AB$, $AD$ keeping these inclined at an angle $\alpha$ to $AC$. Show that the thrust on the stiffening rod is $(4W + 2W') \tan \alpha$.
6. A rhombus$ABCD$ is formed of four equal uniform rods freely jointed together and suspended from the end $A$; it is kept in position by a light rod joining the middle points of $BC$ and $CD$; if $T$ be the thrust in this rod and $W$ the weight of the rhombus, prove $T = W \tan \frac{A}{2}$.
7. The middle points of opposite sides of a quadrilateral formed by four freely jointed weightless bars are connected by two light rods of lengths $a$ and $b$ in a state of tension. If $T_1$ and $T_2$ be the tensions of those rods, prove that $\frac{T_1}{a} + \frac{T_2}{b} = 0$.
8. A heavy uniform rod of length $2a$, rests with its ends in contact with two smooth inclined planes of inclination $\alpha$ and $\beta$ to the horizon. If $\gamma$ be the inclination of the rod to the horizon, prove that by the principle of V.W., $2 \tan \gamma = \cot \alpha - \cot \beta$.
9. A frame consists of five weightless rods forming the sides of a rhombus $ABCD$ with diagonal $AC$. If four equal forces $F$ act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in $AC$ is $\frac{F \cos 2\alpha}{\sin \alpha}$, where $\alpha$ is the angle $BAC$.
10. A solid hemisphere is supported by a string fixed to point on its rim and to a point on a smooth vertical wall with which the curved surface of the sphere is in contact. If $\alpha, \beta$ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that $\tan \beta = \frac{3}{8} + \tan \alpha$.
11. A regular pentagon $ABCDE$ is formed of five uniform heavy rods each of weight $W$freely jointed of their extremities. It is freely suspended from $A$ and is maintained in its regular pentagonal form by a light rod joining $B$ and $E$. Find the stress in this rod.
12. Six equal heavy rods, freely hinged at the ends, form a regular hexagon $ABCDEF$, which hung up by the corner $A$, is kept from altering the shape by two light rods $BF$ and $CE$. Prove that the thrusts in these rods are $\frac{5\sqrt{3}}{2}W$ and $\frac{\sqrt{3}}{2}W$, where $W$ is the weight of each rod.

## Assignment cum Important problems on Forces in Three Dimensions

1. Prove that every system of non-coplanar forces can be reduced into a resultant force $R$ acting at $O$ together with a couple of moment $G$ whose axis passes through $O$.
2. Prove that every system of non-coplanar forces acting on a rigid body can be reduced into a wrench.
3. Determine the conditions that a given system of non-coplanar forces should compounded into a single resultant force.
4. Prove that whatever origin and axes are chosen, for a given system of non-coplanar forces the quantities $X^2 + Y^2 + Z^2$ and $LX + MY + NZ$ are invariants.
5. Define central axis and determine its equation for a given system of non-coplanar forces acting on a rigid body at different points.
6. Determine the equation of central axis when the wrench of the system be reduced into a single force.
7. Three forces each equal to $F$ acting on a body at $(a, 0,0)$ parallel to $Oy$, at $(0, b, 0)$ parallel to $Oz$ and at $(0,0, c)$ parallel to $Ox$, the axes being rectangular. Find the resultant wrench in magnitude and position.
8. $OA, OB, OC$ are three coterminous edges of a cube and $AA', BB', CC'$ and $OO'$ are diagonals. Along $BC', CA', AB'$ and $OO'$ act forces equal to $X, Y, Z$ and $R$ respectively. Show that they are equivalent to a single resultant if $(XY + YZ + ZX)\sqrt{3} + R(X + Y + Z) = 0$.
9. Forces $X, Y, Z$are acting along the three straight lines $y = b, z = -c; z = c, x = -a; x = a, y = -b$, respectively. Find the condition that the system will have a single resultant. Also determine the equation of line of action.
10. Forces $X, Y, Z$are acting along the three straight lines $y = 0, z = c; z = 0, x = a; x = 0, y = b$, respectively. Find the pitch of the equivalent wrench. Also show that the line of action of the force lies on the hyperboloid $(x - a)(y - b)(z - c) = xyz$.
11. Two equal forces act along each of the straight lines $\frac{x \mp a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{\mp b\cos\theta} = \frac{z}{c}$; show that their central axis must, for all $\theta$, lie on the surface $y\left(\frac{x}{z} + \frac{z}{x}\right) = b\left(\frac{a}{c} + \frac{c}{a}\right)$.
12. Two forces $P$ and $Q$ act along the straight lines $y = x\tan\alpha, z = c$ and $y = -x\tan\alpha, z = -c$ respectively. Show that central axis lies on a straight line $y = x\left(\frac{P-Q}{P+Q}\right)\tan\alpha, \frac{z}{c} = \frac{P^2-Q^2}{P^2+2PQ\cos 2\alpha+Q^2}$.For all values of $P$ and $Q$, prove that this line is a generator of the surface $(x^2 + y^2)z\sin 2\alpha = 2cxy$.
13. A force parallel to the axis of $z$ acts at $(a, 0,0)$ and an equal force perpendicular to the axis of $z$ acts at $(-a, 0,0)$. Show that the central axis of the system lies on the surface $(x^2 + y^2)z^2 = (x^2 + y^2 - ax)^2$.

14. Two equal forces act along the generators of the same system of the hyperboloid $\frac{x^2+y^2}{a^2} - \frac{z^2}{b^2} = 1$ and cut the plane $z = 0$ at the extremities of perpendicular diameters of the circles $x^2 + y^2 = a^2$. Find the pitch of the equivalent wrench.

# Centre of Gravity(c.g.)

## 1. Analytical Method

If $w_1, w_2, w_3$, and $w_n$ be the weights of a system of particles placed at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and $(x_n, y_n)$of a two dimensional rigid body then the position of c.g. $G(\overline{x}, \overline{y})$ is given by taking moments about $y$-axis

$$(w_1 + w_2 + w_3 + \cdots + w_n)\overline{x} = w_1x_1 + w_2x_2 + w_3x_3 + \cdots + w_nx_n$$

$$\therefore \quad \overline{x} = \frac{\sum_i w_ix_i}{\sum_i w_i}.$$

Similarly, if we take moments about $y$-axis, then we get

$$(w_1 + w_2 + w_3 + \cdots + w_n)\overline{y} = w_1y_1 + w_2y_2 + w_3y_3 + \cdots + w_ny_n$$

$$\therefore \quad \overline{y} = \frac{\sum_i w_iy_i}{\sum_i w_i}.$$

If $m_1, m_2, m_3$, and $m_n$ be the masses of a system of particles, then weights being proportional to the masses and then we get

$$\overline{x} = \frac{\sum_i m_ix_i}{\sum_i m_i}, \overline{y} = \frac{\sum_i m_iy_i}{\sum_i m_i}.$$

If we are considering a **continuous distribution of matter** in the form of a rigid lamina of any shape then we have

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int xdm}{\int dm}, \overline{y} = \frac{\sum y\delta m}{\sum \delta m} = \frac{\int ydm}{\int dm}.$$

If we are dealing with a **three dimensional body** then the co-ordinates of c.g. are given by

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int xdm}{\int dm}, \overline{y} = \frac{\sum y\delta m}{\sum \delta m} = \frac{\int ydm}{\int dm}, \overline{z} = \frac{\sum z\delta m}{\sum \delta m} = \frac{\int zdm}{\int dm}.$$

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## 2. c.g. of the combined body

If two bodies $A$ and $B$ are rigidly joined together with their weights $W_1$ and $W_2$and if their c.g. be at $G_1$ and $G_2$ w.r.t. the fixed point $O$ then distance of c.g. of the whole body (combined body) from $O$ is given by

$$(W_1 + W_1)OG = W_1.OG_1 + W_2.OG_2$$

$$\therefore \quad OG = \frac{W_1.OG_1 + W_2.OG_2}{W_1 + W_2},$$

where the signs of $OG_1$and $OG_2$have to be taken into account.

## 3. c.g. of the remaining part

If $W$ be the weight of the whole body and $G$ the c.g.and $W_1$be the weight of the removed part then the c.g. of the remaining portion is

$$OG_2 = \frac{W_1.OG - W_1.OG_1}{W - W_1},$$

where the signs of $OG$and $OG_1$have to be taken into account.

## 4. c.g. of an arc

Let $AB$ be an arc of the a plane curve and let us consider an elementary arc $PQ$ at $P$ of length $\delta s$ at an arc lenth $s$ from $A$ and $\rho$ be the mass per unit length. Then $\delta m = \rho\delta s$. If $G(\overline{x}, \overline{y})$ be the position of c.g. then

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int xdm}{\int dm} = \frac{\int \rho xds}{\int \rho ds}, \overline{y} = \frac{\sum y\delta m}{\sum \delta m} = \frac{\int ydm}{\int dm} = \frac{\int \rho yds}{\int \rho ds}$$

where limits extending from $A$ to $B$.

If $\rho$ is constant then

$$\overline{x} = \frac{\int xds}{\int ds}, \overline{y} = \frac{\int yds}{\int ds}.$$

If the curve be of the form

i) $y = f(x)$ then $ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$,

ii) $x = f(y)$ then $ds = \sqrt{1 + (\frac{dx}{dy})^2}\, dy$,

iii) $x = f(t), y = g(t)$ then $ds = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$,

iv) $f(r, \theta) = 0$ then $ds = \sqrt{(r)^2 + (\frac{dr}{d\theta})^2}\, d\theta = \sqrt{1 + (r\frac{d\theta}{dr})^2}\, dr$ with $x = r\cos\theta, y = r\sin\theta$.

## 5. c.g. of a plane area

Let us consider an elementary area $\delta S$ about the point $P(x, y)$ on the area $S$. If $\rho$ be the mass per unit area of the plane curve then $\delta m = \rho\delta S$. If $G(\overline{x}, \overline{y})$ be the position of c.g. then

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int xdm}{\int dm} = \frac{\int \rho xdS}{\int \rho dS}, \overline{y} = \frac{\sum y\delta m}{\sum \delta m} = \frac{\int ydm}{\int dm} = \frac{\int \rho ydS}{\int \rho dS}$$

where limits extending over whole area $S$.

In Cartesian curve $dS = dxdy$, then

$$\overline{x} = \frac{\iint \rho xdxdy}{\iint \rho dxdy}, \overline{y} = \frac{\iint \rho ydxdy}{\iint \rho dxdy}.$$

In Polar curve $dS = rdrd\theta$, then

$$\overline{x} = \frac{\iint \rho xdxdy}{\iint \rho dxdy} = \frac{\iint \rho r^2\cos\theta\, drd\theta}{\iint \rho drd\theta}, \overline{y} = \frac{\iint \rho ydrd\theta}{\iint \rho drd\theta} = \frac{\iint \rho r^2\sin\theta\, drd\theta}{\iint \rho drd\theta}.$$

If $\rho$ is constant then

$$\overline{x} = \frac{\int xdS}{\int dS}, \overline{y} = \frac{\int ydS}{\int dS}.$$

## 6. c.g. of a surface of revolution

Let the plane curve $AB$ revolve around the axis of $x$ and generates a surface. If $PQ$ be an element of arc of length $\delta s$, the corresponding element of the surface f revolution has an area $2\pi y\delta s$ and if $\rho$ be the density of the matter in this portion then $\delta m = \rho 2\pi y\delta s$. If $G(\overline{x}, \overline{y}, \overline{z})$ be the position of c.g. then

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int x\rho 2\pi yds}{\int \rho 2\pi yds} = \frac{\int x\rho yds}{\int \rho yds},$$

$\overline{y} = 0, \overline{z} = 0$, from symmetry of the surface.

## 7. c.g. of a solid of revolution

Let the solid is formed by the revolution of the curve $y = f(x)$ about the axis of $x$ and suppose it is bounded by two ordinates $LA$ and $MB$ with corresponding abscissa. The volume is generated by an elementary circular area $PQP'Q'$ where $(x, y)$ and $(x + \delta x, y + \delta y)$ are the co-ordinates of $P$ and $Q$. Then volume of the area is $\pi y^2\delta x$ and if $\rho$ be the density of the matter in this portion then $\delta m = \rho\pi y^2\delta x$. If $G(\overline{x}, \overline{y}, \overline{z})$ be the position of c.g. then

$$\overline{x} = \frac{\sum x\delta m}{\sum \delta m} = \frac{\int x\rho\pi y^2 dx}{\int \rho\pi y^2 dx} = \frac{\int x\rho y^2 dx}{\int \rho y^2 dx},$$

$\overline{y} = 0, \overline{z} = 0$, from symmetry of the surface. If the solid be of uniform density then

$$\overline{x} = \frac{\int xy^2 dx}{\int y^2 dx}, \overline{y} = 0, \overline{z} = 0.$$

## 8. Theorem of Pappus

If any plane area revolve through any angle about an axis in its own plan then

I) *the volume generated by the area is equal to the product of the area and the length of the path described by the centroid of the area*.

and II) *the surface generated by the area is equal to the product of the perimeter of the area and the length of the path described by the centroid of the perimeter*.

## 9. Lagrange's Theorem

If $G$ be the position of c.g. of a system of particles of masses $m_1, m_2, m_3,$ and $m_n$ situated at the points $A_1, A_2, A_3,$ and $A_n$ and $O$ be any other point then

$$\sum_{i=1}^{n} m_i {OA_i}^2 = \sum_{i=1}^{n} m_i {GA_i}^2 + M.OG^2 ,$$

Where $M = m_1 + m_2 + m_3 + \cdots + m_n = \sum_{i=1}^{n} m_i$.

## Assignment cum Important problems on c.g.

1. Find the position of c.g. of a circular arc making an angle $2\alpha$ at the centre.
2. Find the position of c.g. of a segment of a circular disc making an angle $2\alpha$ at the centre. Hence determine the position of c.g. of a semicircular disc.
3. Find the position of c.g. of the area bounded by the axis of $y$ and the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$.
4. Find the position of c.g. of the arc of the cardioids $r = a(1 + \cos\theta)$ lying above the initial line. What happen in case of area of the cardioids?
5. Find the position of c.g. of the area included between the curve $y^2(2a - x) = x^3$ and its asymptote.
6. Find the position of c.g. of the area enclosed by the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$ lying in the first quadrant.
7. If the density of a circular arc varies as the square of the distance from a point $O$ on the arc, show that its c.g. divides the diameter through $O$ in the ratio $3:1$.
8. Find the c.g. of the surface generated by the revolution of a loop of the lemniscates $r^2 = a^2\cos 2\theta$, about the initial line.
9. Find the centroid of the surface formed by the revolution of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ about the axis of $y$.
10. Find the position of c.g. of a lamina in the shape of a quadrant of the curve $(\frac{x}{a})^{2/3} + (\frac{y}{b})^{2/3} = 1$, density being given by $\rho = kxy$.
11. An isosceles triangular lamina is such that its mass per unit area at every point is proportional to the sum of the distances of the point from the equal sides of the triangle. Prove that the distance of the c.g. from the vertex is three-fourth of the altitude.
12. An isosceles triangular lamina is such that its mass per unit area at every point is proportional to the product of the distances of the point from the equal sides of the triangle. Prove that the distance of the c.g. from the vertex is fourth-fifth of the altitude.
13. Find the c.g. of a hemisphere whose density varies as the distance from a point on its plane edge.
14. A circular disc of radius $a$ whose density is proportional to the distance from the centre, has a hole cut in it bounded by a circle of diameter $b$ which passes through the centre. Show that the distance from the centre of the disc, of c.g. of the remaining portion is $\frac{6b^4}{15\pi a^3 - 10b^3}$.
15. A frustum of a uniform right circular cone whose semi vertical angle is $\alpha$ is made by cutting of $\frac{1}{n}$th of the axis. Prove that the frustum will rest with a slant side on a horizontal plane if $\tan^2\alpha < \frac{3n^4 - 4n^3 + 1}{4(n^3 - 1)}$.

## Reference Books:

**1. Analytical Statics, M. C. Ghosh**

**2. Advanced Analytical Statics, S. Mondal**